M251 Practice Exam for 9.1-9.6

For every problem that states "determine the convergence or divergence of the series" also ***justify your answer by identifying the theorem or test and showing how the condition or conditions were satisfied.***

1. Write the first five terms of the sequence.

$$a_n = \left(-\frac{4}{5}\right)^n$$

$\frac{-4}{5}, \frac{16}{25}, \frac{-64}{125}, \frac{256}{625}, \frac{-1024}{3125}$

2. Write the first five terms of the sequence.

$$a_n = (-1)^{n+4}\left(\frac{17}{n}\right)$$

$-17, \frac{17}{2}, \frac{-17}{3}, \frac{17}{4}, \frac{-17}{5}$

3. Write the first five terms of the sequence.

$$a_n = 5 - \frac{3}{n} - \frac{7}{n^2}$$

$-5, \frac{7}{4}, \frac{29}{9}, \frac{61}{16}, \frac{103}{25}$

4. Determine the convergence or divergence of the sequence with the given nth term. If the sequence converges, find its limit.

$$a_n = \frac{\ln(n^{10})}{6n}$$

$\frac{\infty}{\infty}$ form $\rightarrow$ L'Hopital's rule

$= \boxed{\lim_{n\to\infty} \frac{10\frac{1}{n}}{6} = 0 \text{ converges}}$

5. Determine the convergence or divergence of the sequence with the given nth term. If the sequence converges, find its limit.

$$a_n = \frac{\ln\left(\sqrt[9]{n}\right)}{8n} = \frac{\ln n}{9 \cdot 8n}$$ $\frac{\infty}{\infty}$ form, by L'Hopital's Rule:

$$\lim a_n = \lim_{n\to\infty} \frac{\frac{1}{n}}{72} = 0$$

6. Determine the convergence or divergence of the sequence with the given nth term. If the sequence converges, find its limit.

$$a_n = \frac{2^n}{5^n} = \left(\frac{2}{5}\right)^n \longrightarrow 0$$

7. Write the first five terms of the sequence of partial sums.

$$5+\frac{5}{4}+\frac{5}{9}+\frac{5}{16}+\frac{5}{25}+\cdots$$

$S_1 = 5$
$S_2 = 5 + \frac{5}{4} = 6.25$
$S_3 = S_2 + \frac{5}{9} = \frac{245}{36}$
$S_4 = \frac{1025}{144}$
$S_5 = \frac{5269}{720}$

8. Write the first five terms of the sequence of partial sums.

$$-5+\frac{25}{6}-\frac{125}{36}+\frac{625}{216}-\frac{3125}{1296}+\cdots$$

$S_1 = -5$
$S_2 = -5/6$
$S_3 = -155/36$
$S_4 = -305/216$
$S_5 = -4955/1296$

9. Write the first five terms of the sequence of partial sums.

$$\sum_{n=1}^{\infty} \frac{5}{(4)^{n-1}}$$

$S_1 = 5$
$S_2 = S_1 + \frac{5}{4} = 6.25$
$S_3 = S_2 + \frac{5}{16} = \frac{105}{16}$
$S_4 = S_3 + \frac{5}{64} = 425/64$
$S_5 = S_4 + \frac{5}{256} = 1705/256$

10. Find the sum of the convergent series.

$$\sum_{n=1}^{\infty} \frac{6}{(n+4)(n+6)}$$

$$\frac{6}{(n+4)(n+6)} = \frac{A}{n+4} + \frac{B}{n+6}$$

$$6 = (n+6)A + (n+4)B$$

$n=-4 \Rightarrow 6 = 2A \Rightarrow 3 = A$

$n=-6 \Rightarrow 6 = -2B \Rightarrow -3 = B$

$3\sum_{n=1}^{\infty} \frac{1}{n+4} - \frac{1}{n+6}$ is a telescoping series

$$3\left(\frac{1}{5} - \frac{1}{7} + \frac{1}{6} - \frac{1}{8} + \frac{1}{7} - \frac{1}{9} + \frac{1}{8} - \frac{1}{10} + \dots\right.$$

$$S_n = 3\left(\frac{1}{5} + \frac{1}{6} - \frac{1}{n+5} - \frac{1}{n+6}\right) \rightarrow \frac{33}{30} = \frac{11}{10}$$

11. Find the sum of the convergent series.

$$\sum_{n=1}^{\infty} (-1)^n \frac{4}{(n+9)(n+11)}$$

$$\frac{4}{(n+9)(n+11)} = \frac{A}{n+9} + \frac{B}{n+11}$$

$$4 = (n+11)A + (n+9)B$$

$n=-9 \Rightarrow 4 = 2A \Rightarrow 2 = A$

$n=-11 \Rightarrow 4 = -2B \Rightarrow -2 = B$

$2\sum_{n=1}^{\infty} \frac{(-1)^n}{n+9} - \frac{(-1)^n}{n+11}$ is a telescoping series

$$2\left(\frac{-1}{10} + \frac{1}{12} + \frac{1}{11} - \frac{1}{13} - \frac{1}{12} + \frac{1}{14} + \frac{1}{13} - \frac{1}{15} - \frac{1}{14} + \frac{1}{16} + \dots\right.$$

$$S_n = 2\left(\frac{-1}{10} + \frac{1}{11} \pm \frac{1}{n+10} \pm \frac{1}{n+11}\right) \rightarrow 2\left(\frac{-1}{10} + \frac{1}{11}\right) = \frac{-1}{55}$$

12. Find the sum of the convergent series.

$\sum_{n=0}^{\infty} 9\left(\frac{10}{11}\right)^n$ geometric series with $r = \frac{10}{11} < 1$

converges to $\frac{\text{first}}{1-\text{ratio}} = \frac{9}{1-\frac{10}{11}} = 99$

13. Find the sum of the convergent series.

$\sum_{n=0}^{\infty} 2\left(-\frac{9}{10}\right)^n$ geometric series with $-1 < r = -\frac{9}{10} < 1$

converges to $\frac{\text{first}}{1-\text{ratio}} = \frac{2}{1-\left(-\frac{9}{10}\right)} = \frac{20}{19}$

14. Determine the convergence or divergence of the series.

$\sum_{n=1}^{\infty} \frac{4^{-n}}{n^{-4}} = \sum_{n=1}^{\infty} \frac{n^4}{4^n}$ using the ratio test we have

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{\frac{(n+1)^4}{4^{n+1}}}{\frac{n^4}{4^n}} = \lim_{n\to\infty} \left(\frac{n+1}{n}\right)^4 \frac{4^n}{4^{n+1}} = \frac{1}{4}$$

so the series converges

15. Determine the convergence or divergence of the series.

$$\sum_{n=0}^{\infty} \frac{2}{2^n}$$

This is a geometric series with $r = \frac{1}{2} \Rightarrow |r| < 1 \Rightarrow$ series converges

16. Find all values of x for which the series converges. For these values of x, write the sum of the series as a function of x.

$$\sum_{n=0}^{\infty} \frac{x^n}{9^n}$$

This is a geometric series with $r = \frac{x}{9}$ which converges for $-1 < \frac{x}{9} < 1$ ie $-9 < x < 9$. For those values of x the series converges to $\frac{\text{first}}{1-\text{ratio}} = \frac{1}{1-\frac{x}{9}} = \frac{9}{9-x}$

17. Find all values of x for which the series converges. For these values of x, write the sum of the series as a function of x.

$$\sum_{n=0}^{\infty} 10\left(\frac{x-4}{10}\right)^n$$

This is a geometric series with $r = \frac{x-4}{10}$ which converges for $-1 < \frac{x-4}{10} < 1$ $\Rightarrow -10 < x-4 < 10 \Rightarrow -6 < x < 14$. For those values the series converges to $\frac{\text{first}}{1-\text{ratio}} = \frac{10}{1-\frac{x-4}{10}} = \frac{100}{14-x}$

18. Use the Integral Test to determine the convergence or divergence of the series. Show your work.

$$\sum_{n=1}^{\infty} \frac{7}{10n+2}$$

Let $f(x) = \frac{7}{10x+2}$ then $f'(x) = \frac{-70}{(10x+2)^2} < 0$ so $f(x)$ is differentiable, continuous, and decreasing $\Rightarrow$ series and $\int_1^\infty f(x)dx$ converge or diverge together.

$\int_1^\infty \frac{7}{10x+2} dx = \lim_{b \to \infty} \left[\frac{7}{10} \ln(10x+2)\right]_1^b = \frac{7}{10}\left(\lim_{b\to\infty} (10b+2) - \ln 12\right) = \infty$

so the series diverges

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19. Use the Integral Test to determine the convergence or divergence of the series. Show your work.

$$\sum_{n=1}^{\infty} ne^{-\frac{n}{2}}$$

Let $f(x)=xe^{-\frac{x}{2}}$ then $f'(x)=e^{-\frac{x}{2}}-\frac{x}{2}e^{-\frac{x}{2}}=(1-\frac{x}{2})e^{-\frac{x}{2}}<0$

so $f(x)$ is continuous and decreasing.

Thus by the integral test the series converges or diverges with $\int_1^{\infty} xe^{-\frac{x}{2}}dx$ integrating by parts $u=x$ $dv=e^{-\frac{x}{2}}dx$

$du=dx$ $v=-2e^{-\frac{x}{2}}$

$\lim_{b\to\infty}\left[-2xe^{-\frac{x}{2}}-\int -2e^{-\frac{x}{2}}dx\right]_1^b \to \lim_{b\to\infty}\left[-2xe^{-\frac{x}{2}}-4e^{-\frac{x}{2}}\right]_1^b$

$\left[\frac{-2(x+2)}{e^{\frac{x}{2}}}\right]_1^b$

use L'Hopital's rule once to get $\frac{-2}{\frac{1}{2}e^{\frac{x}{2}}}\to 0$

So the series converges

20. Use the Integral Test to determine the convergence or divergence of the series. Show your work.

$$\sum_{n=2}^{\infty}\frac{\ln n}{n^7}$$

Let $f(x)=\frac{\ln x}{x^7}$, $f'(x)=\frac{\frac{1}{x}x^7-7x^6\ln x}{x^{14}}=\frac{1-7\ln x}{x^8}<0$

so $f(x)$ is continuous and decreasing.

$\int\frac{\ln x}{x^7}dx$ integrating by parts $u=\ln x$ $dv=\frac{dx}{x^7}$ $du=\frac{dx}{x}$ $v=\frac{1}{-6x^6}$

$=-\frac{\ln x}{6x^6}-\int\frac{dx}{-6x^7}=-\frac{\ln x}{6x^6}-\frac{1}{36x^6}$ For the limit use L'Hopital's rule

21. Use the Integral Test to determine the convergence or divergence of the series. Show your work.

$$\sum_{n=2}^{\infty}\frac{4}{n\sqrt{\ln n}}$$

$f(x)=\frac{4}{x\sqrt{\ln x}}$ is continuous and decreasing.

$\int\frac{4}{x\sqrt{\ln x}}dx$ substitute $u=\ln x$, $du=\frac{dx}{x}$

$\int\frac{4}{\sqrt{u}}du = 8\sqrt{u}=8\sqrt{\ln x}$; $\int_2^{\infty}\frac{4}{x\sqrt{\ln x}}dx=\lim_{b\to\infty}\left[8\sqrt{\ln x}\right]_2^{\infty}\to\infty$

so the series diverges

22. Use the p-series theorem to determine the convergence or divergence of the series.

$\sum_{n=1}^{\infty}\frac{8}{n^{\frac{10}{7}}}=8\sum_{n=1}^{\infty}\frac{1}{n^{\frac{10}{7}}}$ which is a p-series with $p=\frac{10}{7}>1$

so the series converges.

23. Use the p-series theorem to determine the convergence or divergence of the series.

$1+\frac{1}{\sqrt[3]{2^2}}+\frac{1}{\sqrt[3]{3^2}}+\frac{1}{\sqrt[3]{4^2}}+\frac{1}{\sqrt[3]{5^2}}+\cdots=\sum_{n=1}^{\infty}\frac{1}{n^{\frac{2}{3}}}$ which is a p-series

with $p=\frac{2}{3}<1$ so the series diverges

24. Use the p-series theorem to determine the convergence or divergence of the series.

$$\sum_{n=1}^{\infty} \frac{1}{n^{0.78}}$$ This is a p series with $p = 0.78 < 1$ so the series diverges

25. Determine the convergence or divergence of the series.

$$\sum_{n=1}^{\infty} \frac{1}{4n^2 - 1}$$

soln 1: clearly $1 < 3n^2 \Rightarrow 0 < 3n^2 - 1 \Rightarrow n^2 < 4n^2 - 1$

$\Rightarrow \frac{1}{4n^2-1} < \frac{1}{n^2}$. $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is a convergent p-series ($p = 2 > 1$) so by D.C.T. $\sum \frac{1}{4n^2-1}$ converges

soln 2: using the LCT with $\sum \frac{1}{n^2}$ $\lim_{n\to\infty} \frac{\frac{1}{n^2}}{\frac{1}{4n^2-1}} = \lim \frac{4n^2-1}{n^2} = \lim 4 - \frac{1}{n^2} = 4.$

$\sum \frac{1}{n^2}$ converges (p series w $p = 2 > 1$) so $\sum \frac{1}{4n^2-1}$ converges

26. Determine the convergence or divergence of the series.

$$\sum_{n=1}^{\infty} \frac{7}{n \cdot \sqrt[8]{n}} = 7 \sum_{n=1}^{\infty} \frac{1}{n^{1.125}}$$ which is a pseries with $p = \frac{9}{8} > 1$ so the series converges

27. Determine the convergence or divergence of the series.

$$3 \cdot \sum_{n=1}^{\infty} \frac{1}{n^{0.95}}$$ this is a p-series with $p = 0.95 < 1$ so the series diverges.

28. Determine the convergence or divergence of the series.

$$\sum_{n=0}^{\infty}\left(\frac{5}{3}\right)^n$$

This is a geometric series with $r=\frac{5}{3}>1$ so the series diverges.

29. Use the Ratio Test to determine the convergence or divergence of the series.

$$\sum_{n=1}^{\infty} n\left(\frac{3}{4}\right)^n$$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{(n+1)\left(\frac{3}{4}\right)^{n+1}}{n\left(\frac{3}{4}\right)^n} = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)\frac{3}{4} = \frac{3}{4}$$

$\frac{3}{4}<1 \Rightarrow$ series converges

30. Use the Ratio Test to determine the convergence or divergence of the series.

$$\sum_{n=1}^{\infty}\frac{n^9}{4^{-n}}$$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{(n+1)^9\, 4^{n+1}}{n^9\, 4^n} = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^9 4 = 4$$

$4>1$ so the series diverges

31. Use the Ratio Test to determine the convergence or divergence of the series.

$$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}\left(\frac{10}{8}\right)^n}{n^2}$$

$$\lim_{n\to\infty}\left|\frac{(-1)^n\left(\frac{10}{8}\right)^{n+1}}{(n+1)^2}\Big/\frac{(-1)^{n-1}\left(\frac{10}{8}\right)^n}{n^2}\right| = \lim_{n\to\infty}\left(\frac{n}{n+1}\right)^2\frac{10}{8} = \frac{10}{8}$$

$\frac{10}{8}>1 \Rightarrow$ series diverges

32. Use the Root Test to determine the convergence or divergence of the series.

$$\sum_{n=1}^{\infty}\left(\frac{8n}{8n+1}\right)^n$$

$$\lim_{n\to\infty}\sqrt[n]{|a_n|} = \lim_{n\to\infty}\left(\frac{8n}{8n+1}\right) = \lim_{n\to\infty}\left(\frac{1}{1+\frac{1}{8n}}\right) = 1$$

so the Root test is inconclusive.

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33. Use the Root Test to determine the convergence or divergence of the series.

$$\sum_{n=1}^{\infty}\left(\frac{4n+1}{8n-1}\right)^n \quad \lim_{n\to\infty}\sqrt[n]{|a_n|} = \lim_{n\to\infty}\frac{4n+1}{8n-1} = \lim_{n\to\infty}\frac{4+\frac{1}{n}}{8-\frac{1}{n}} = \frac{1}{2} < 1$$

$\Rightarrow$ series converges

34. Use the Root Test to determine the convergence or divergence of the series.

$$\sum_{n=1}^{\infty}\left(\frac{7n^2+1}{4n^2-1}\right)^n \quad \lim_{n\to\infty}\sqrt[n]{|a_n|} = \lim_{n\to\infty}\frac{7n^2+1}{4n^2-1} = \lim_{n\to\infty}\frac{7+\frac{1}{n^2}}{4-\frac{1}{n^2}} = \frac{7}{4} > 1$$

$\Rightarrow$ series diverges

35. Determine the convergence or divergence of the series using any appropriate test from this chapter. Identify the test used.

$$\sum_{n=1}^{\infty}\frac{(-1)^n 8}{3n}$$

From the alternating series test

If $f(x) = \frac{8}{3x}$ then $f'(x) = \frac{-8}{3x^2} < 0$

$f' < 0 \Rightarrow$ terms are decreasing; $\lim_{n\to\infty}\frac{8}{3n} = 0$

$\therefore$ this series converges by the A.S.T.

36. Determine the convergence or divergence of the series using any appropriate test from this chapter. Identify the test used.

$$\sum_{n=1}^{\infty}\frac{6}{n^2} = 6\sum_{n=1}^{\infty}\frac{1}{n^2}$$

which is a p-series with $p = 2 > 1$ hence it converges by p-series test

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or using the integral test $f(x) = \frac{6}{x^2}$ is diff. and decreasing on $x \geq 1$. Furthermore

$$\int_1^{\infty}\frac{6}{x^2}\,dx = \lim_{b\to\infty}\left[\frac{-6}{x}\right]_1^b = 6$$

so the series converges.